# Announcements

1) Quiz Thursday, covers 11.1 - 11.3

(EC due)

2) Exam Thursday next week,

covers 7.3, 7.4, 7.8, 11.1 - 11.3,

11.6

## Back to series

We know geometric series.

$$\sum_{n=1}^{\infty} a(r^n) = \begin{cases} \frac{ar}{1-r}, & |r|<1 \\ \text{divergent}, & |r| \geq 1 \end{cases}$$

Today, look at series that

are not geometric

Example 1: $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ NOT geometric.

We have to find the partial sums

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+3n+2}.$$

Before finding the sums, use partial fractions on

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+2)(n+1)}$$

So there are numbers

$A$ and $B$ with

$$\frac{1}{(n+2)(n+1)} = \frac{A}{n+1} + \frac{B}{n+2}$$

We get

$$1 = A(n+2) + B(n+1)$$

let $n = -2$,

$$1 = B(-1), \quad \boxed{B = -1}$$

let $n = -1$,

$$\boxed{1 = A}$$

Then $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$

Write down $S_k = \sum_{n=1}^{k} \left(\underbrace{\frac{1}{n+1} - \frac{1}{n+2}}_{n=1}\right)$

$S_1 = 1^{st}$ term of sequence $= \frac{1}{2} - \frac{1}{3}$

$S_2 =$ sum of the $1^{st}$ 2 terms

$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$

$= \frac{1}{2} - \frac{1}{4}$

$S_3$ = sum of $1^{st}$ 3 terms

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

$$= \frac{1}{2} - \frac{1}{5}$$

$$S_k = \frac{1}{2} - \frac{1}{k+2}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(\frac{1}{2} - \frac{1}{k+2}\right)$$

$$= \frac{1}{2} - \lim_{k\to\infty} \frac{1}{k+2}$$

$$= \boxed{\frac{1}{2}} = \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$$

**Note:** For every sequence $(a_n)_{n=1}^{\infty}$ we've met where $\sum_{n=1}^{\infty} a_n$ converges,

$$\lim_{n \to \infty} a_n = 0.$$ In fact, this is true for <u>every</u> convergent series.

# Test for Divergence:

(inverse of previous page)

If $\boxed{\lim_{n \to \infty} a_n \neq 0}$ (includes limit does not exist), then

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

## Example 2:

Does $\sum_{n=3}^{\infty} \cos\left(\frac{1}{n}\right)$

converge or diverge?

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right)$$

$$= \cos(0)$$

$$= 1 \neq 0$$

By the test for divergence,

$\sum_{n=3}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges.

Question: Is it true that

$\lim_{n\to\infty} a_n = 0$, then

$\sum_{n=1}^{\infty} a_n$ converges?

NO. Here's an example

Example 3: $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

Observe $\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$

$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$

$= \ln(1) = 0.$

Next, we get that

$$\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$$

Then

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left(\ln(n+1) - \ln(n)\right)$$

Compute partial sums

$$S_k = \sum_{n=1}^{k} \left(\ln(n+1) - \ln(n)\right)$$

$$S_1 = \ln(2) - \ln(1) = \ln(2)$$

$$S_2 = \left(\cancel{\ln(2)}\right) + \left(\ln(3) - \cancel{\ln(2)}\right)$$

$$= \ln(3)$$

$$S_3 = \cancel{\ln(2)} + (\cancel{\ln(3)} - \cancel{\ln(2)}) + (\ln(4) - \cancel{\ln(3)})$$

$$= \ln(4)$$

$$S_k = \ln(k+1)$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1)$$

$$= \boxed{\infty}$$

This means $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

diverges.

In summary:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If $\lim_{n\to\infty} a_n = 0$, YOU KNOW NOTHING about $\sum_{n=1}^{\infty} a_n$

What about something like $\sum_{n=1}^{\infty} \frac{1}{n}$ ?  $\lim_{n\to\infty} \frac{1}{n} = 0$

No really good pattern for partial sums.

In fact, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

We'll show this after our break

Note: On a quiz or an exam in the next 2 weeks, if you are asked to "find the sum of a series if it converges", you do one of 2 things:

1) If the series is geometric, use the formula for geometric series

2) If not, find a formula for partial sums, take limit.

# Rules for Convergent Series

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge. Then

1) $$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

( you can distribute $\sum$ )

2) If $c$ is any real number,

$$\sum_{n=1}^{\infty} c(a_n) = c \sum_{n=1}^{\infty} a_n$$

WARNING

You can't distribute

$\sum$ over divergent

series!

# The Integral Test

(Section 11.3)

Remember how we figured out sequential limits:

Given $\lim_{n\to\infty} a_n$, if there is a function $f$ defined on $[1,\infty)$ with $f(n) = a_n$,

then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

if the latter limit exists.

The integral of the function takes the place of the sum of the sequence.

Think of $\sum_{n=1}^{\infty} a_n$ like $\int_1^{\infty} f(x)\,dx$,

if the integral converges (diverges), then the series converges (diverges).

Integral Test Suppose $a_n \geq 0$

for all counting numbers $n$.

Suppose that $a_n = f(n)$

where $f(x)$ is defined on $[1, \infty)$

and is

a) continuous

b) decreasing

Then

1) $\sum_{n=1}^{\infty} a_n$ converges if

$\int_1^{\infty} f(x)$ converges.

2) $\sum_{n=1}^{\infty} a_n$ diverges if

$\int_1^{\infty} f(x)dx$ diverges.

Points:

1) The integral test works both ways; if series converges (diverges) then the integral converges (diverges)

2) The starting point $n=1$ is unimportant. You can pick a number bigger than one and still use the test. Start integral where you start the series

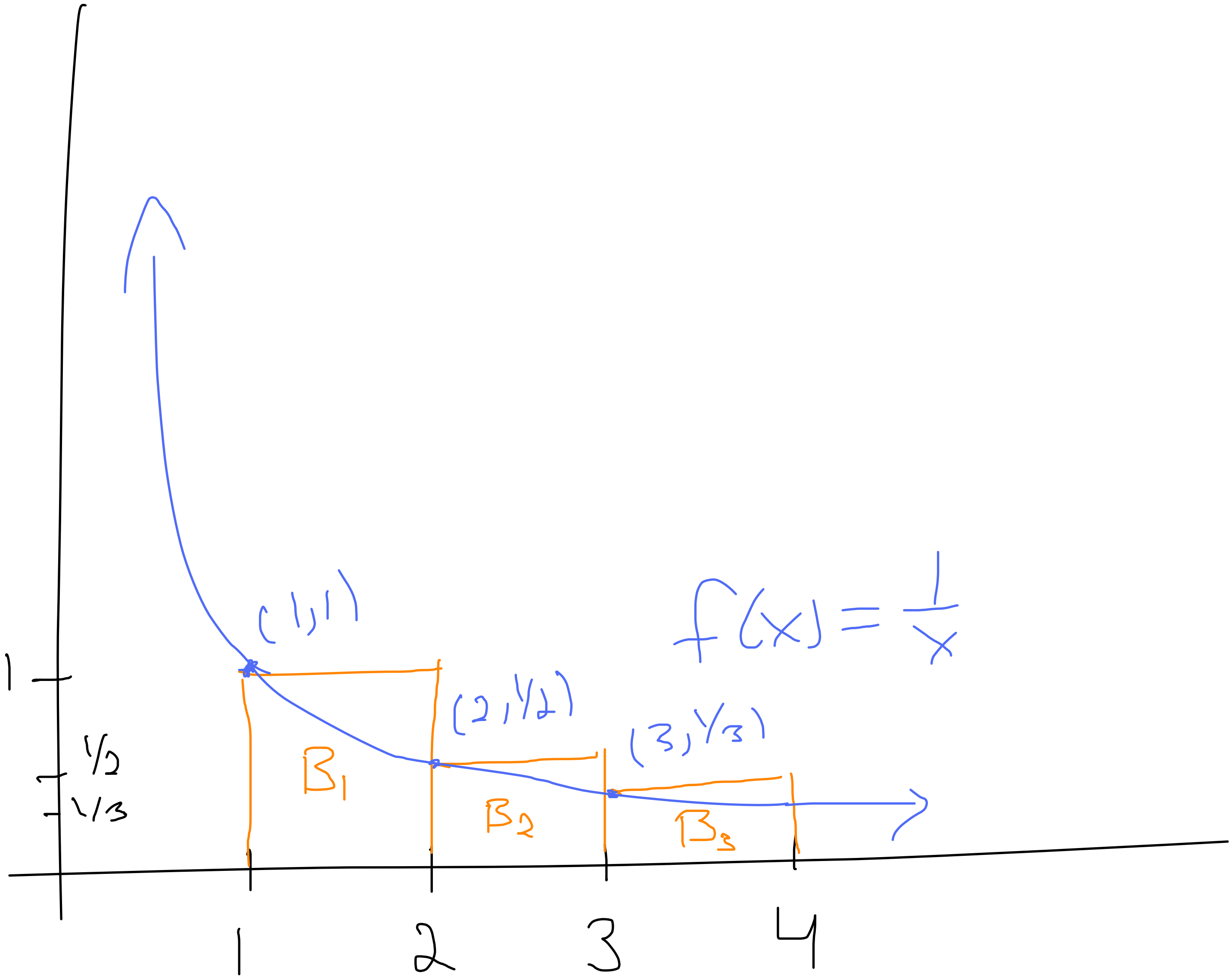
Example 4: $\sum_{n=1}^{\infty} \frac{1}{n}$

$f(x) = \frac{1}{x}$ satisfies

$f(n) = \frac{1}{n}$.

Look at $\int_1^{\infty} \frac{1}{x}\,dx$

$1^{st}$, a picture




Height of $B_n = \frac{1}{n}$

From picture:

$$\int_n^{n+1} \frac{1}{x}\,dx < \text{Area of } B_n$$
$$= \frac{1}{n}.$$

So:

$$\sum_{n=1}^{k} \frac{1}{n} > \sum_{n=1}^{k}\left(\int_n^{n+1} \frac{1}{x}\,dx\right)$$
$$= \int_1^2 \frac{1}{x}\,dx + \int_2^3 \frac{1}{x}\,dx + \int_3^4 \frac{1}{x}\,dx$$
$$+ \cdots + \int_k^{k+1} \frac{1}{x}\,dx$$

Can combine the integrals as

$$\int_1^{k+1} \frac{1}{x}\,dx < \sum_{n=1}^{k} \frac{1}{n}$$

But $$\int_1^{k+1} \frac{1}{x}\,dx = \ln(x) \Big|_1^{k+1}$$

$$= \ln(k+1).$$

Then

$$\ln(k+1) < \sum_{n=1}^{k} \frac{1}{n},$$

so taking limit as $k \to \infty$,

$$\lim_{k\to\infty} \ln(k+1) \leq \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\shortparallel$$

$$\infty$$

This says

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$ so diverges.

Example 5 : $\sum_{n=4}^{\infty} \frac{1}{n^2}$ ,

Use integral test, find

$$\int_4^{\infty} \frac{1}{x^2}\, dx$$

$$= \lim_{t \to \infty} \int_4^{t} \frac{1}{x^2}\, dx$$

$$= \lim_{t \to \infty} \int_4^{t} x^{-2}\, dx$$

$$= \lim_{t \to \infty} \left( -\frac{1}{x} \right) \Big|_4^{t}$$

Then

$$\lim_{t\to\infty} \left(-\frac{1}{x}\right)\Big|_4^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{4}\right)$$

$$= \frac{1}{4} < \infty, \text{ so}$$

the integral converges, which tells us the series $\boxed{\text{converges}}$

# p-rule for series

Let $k$ be a counting number.

Then

$$\sum_{k=n}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges if } p>1 \\ \\ \text{diverges if } p\leq 1 \end{cases}$$

(comes from p-rule for integrals)

Quick example:

$$\sum_{n=22}^{\infty} \frac{1}{n^{\pi}}$$ converges since $\pi > 1$

What's the sum?

I don't know.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

I think $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Extra Credit: Find such a nice formula for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^7}$.

Worth an A in the class!

We shift to only asking whether a series converges or diverges, and won't worry about the sum.

Example 6:

$$\sum_{n=12}^{\infty} \frac{n^2}{e^n}$$

Associated integral is

$$\int_{12}^{\infty} \frac{x^2}{e^x}\,dx = \int_{12}^{\infty} x^2 e^{-x}\,dx$$

$$= \lim_{t\to\infty} \int_{12}^{t} x^2 e^{-x}\,dx$$

$$\int_{12}^{t} x^2 e^{-x}\,dx$$

Tabular Method

| $u$ | | $dv$ |
|---|---|---|
| $x^2$ | $+$ | $e^{-x}$ |
| $2x$ | $-$ | $-e^{-x}$ |
| $2$ | $+$ | $e^{-x}$ |
| $0$ | | $-e^{-x}$ |

So $$\int_{12}^{t} x^2 e^{-x}\,dx = \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\Big|_{12}^{t}$$

$$\left(-x^2e^{-x}-2xe^{-x}-2e^{-x}\right)\Big|_{12}^{t}$$

$$= -e^{-x}\left(x^2+2x+2\right)\Big|_{12}^{t}$$

$$= -\frac{\left(x^2+2x+2\right)}{e^x}\Big|_{12}^{t}$$

$$= -\frac{\left(t^2+2t+2\right)}{e^t} + \frac{170}{e^{12}}$$

take limit as $t \to \infty$!

$$\lim_{t\to\infty}\left(-\frac{(t^2+2t+2)}{e^t}+\frac{170}{e^{12}}\right)$$

$$=\left(\underbrace{-\lim_{t\to\infty}\frac{t^2+2t+2}{e^t}}_{}\right)+\frac{170}{e^{12}}$$

$$\overset{\text{l'H}}{=}\lim_{t\to\infty}\frac{2t+2}{e^t}$$

$$\overset{\text{l'H}}{=}\lim_{t\to\infty}\frac{2}{e^t}=0$$

So $\displaystyle\int_{12}^{\infty} \frac{x^2}{e^x}\,dx = \frac{170}{e^{12}} < \infty$,

so converges, which

says

$$\sum_{n=12}^{\infty} \frac{n^2}{e^n}$$ $\boxed{\text{converges}}$ by

the integral test.